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                COMPONENT-WISE MINIMAX PROPERTY OF STEIN'S RULE.(U)
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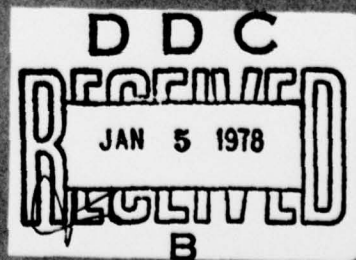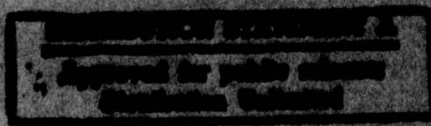UNCLASSIFIED   N87   NL

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CLEMSON UNIVERSITY
Clemson, South Carolina

COMPONENT-WISE MINIMAX

PROPERTY OF STEIN'S RULE

BY

KHURSHEED ALAM

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TECHNICAL REPORT #255

Component-Wise Minimax Property of Stein's Rule

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ABSTRACT

For estimating the mean of a p-variate normal distribution a family of estimators known as Stein's estimators are known to dominate the maximum likelihood estimator and are therefore minimax when $p \geq 3$, under quadratic loss, equal to the sum of squared errors of the components of the estimator. It is shown in this paper for a subfamily, the vector consisting of any K components of the estimator is also minimax for estimating the corresponding components of the mean vector, where $3 \leq K \leq p$.

1. **Introduction.** Let the p-component vector $X = (X_1,\ldots,X_p)'$ be normally distributed with mean $\theta = (\theta_1,\ldots,\theta_K)'$ and convariance $\sigma^2 I$ where $p \geq 3$, I denotes the identity matrix and σ is known. For estimating θ let the loss be equal to the sum of squared errors, given by

$$L(\delta,\theta) = \sum_{i=1}^{p} (\delta_i - \theta_i)^2$$

where $\delta(X) = (\delta_1(X),\ldots,\delta_p(X))$ is an estimate of θ, based on X. The maximum likelihood estimator X is minimax but inadmissible for $p \geq 3$ with respect to the given loss function. The inadmissibility of X was first proved by Stein (1955). An estimator which dominates X and is known as Stein's estimator, is given by

$$(1.1) \qquad \delta^*(X) = (1 - \frac{\nu(p-2)\sigma^2}{S}) X$$

where $S = \sum_{i=1}^{p} X_i^2$ and $0 < \nu < 2$. The risk of δ^* is given by

$$(1.2) \qquad R(\delta^*,\theta) = E\ L(\delta^*,\theta)$$
$$= p\sigma^2(1 - \nu(2-\nu)(p-2)^2 E(\frac{\sigma^2}{pS})).$$
$$< p\sigma^2 = R(X,\theta).$$

We are interested in assessing the performance of δ^* component-wise. Let

$$(1.3) \qquad R_i^* = E(\delta_i^* - \theta_i)^2$$

denote the contribution to the risk of δ^* from the ith component. It is shown in the following section that for any integer K, such that $3 \leq K \leq p$

(1.4)
$$\sum_{i=1}^{K} R_i^* < K\sigma^2$$

for $0 < \nu < \dfrac{2(K-2)}{p-2}$. The above inequality shows that δ^* is minimax for estimating three or more components of θ. We also generalize the above result for the case in which the variances of the components of X are different but known or unknown up to a constant factor. The result is extended to another class of estimators.

2. <u>Main results</u>. We shall compute the value of R_i^*, the contribution to the risk of δ^* from the ith component. First we consider the case where σ is known. We let $\sigma = 1$, without loss of generality. Let $f_m(y)$ denote the density function of a chi-square random variable with m degrees of freedom and

$$g(x) = (2\pi)^{-p/2} \exp(-\frac{1}{2}(X-\theta)'(X-\theta))$$

denote the normal density function. For any integrable function ϕ, we have

(2.1)
$$\begin{aligned}
E \, X_i\phi(X'X) &= e^{-\theta_i^2/2} \frac{\partial}{\partial \theta_i} (e^{\theta_i^2/2} \int \phi(x'x)g(x)dx) \\
&= \theta_i \, e^{-\theta'\theta/2} \sum_{r=1}^{\infty} \frac{(\theta'\theta/2)^{r-1}}{(r-1)!} \int_0^\infty \phi(y)f_{p+2r}(y)dy \\
&= \theta_i E\phi(\chi_{p+2,\theta'\theta}^2)
\end{aligned}$$

where $\chi^2_{m,\lambda}$ denotes a non-central chi-square random variable with m degrees of freedom and noncentrality parameter equal to λ. Similarly

$$(2.2) \qquad E\ X^2_i \phi(X'X) = E\ \phi(X^2_{p+2,\theta'\theta}) + \theta^2_i\ E\ \phi(X^2_{p+4,\theta'\theta}).$$

Let $\lambda = \theta'\theta$. We have $E(\chi^2_{p+2,\lambda})^{-1} = \frac{1}{p}\ e^{-\lambda/2}\phi(\frac{p}{2},\ \frac{p}{2}+1;\ \frac{\lambda}{2})$ where

$$\phi(a,b;y) = 1 + \frac{a}{b}\ y + \frac{a(a+1)}{b(b+1)}\cdot\frac{y^2}{2!} + \cdots$$

denotes the confluent hypergeometric function. From (1.3), (2.1) and (2.2) we get by direct computation

$$(2.3) \quad R^*_i = (1-\frac{\nu(p-2)}{\chi^2_{p+2,\lambda}})^2 + \theta^2_i[(1-\frac{\nu(p-2)}{\chi^2_{p+4,\lambda}})^2 - 2(1-\frac{\nu(p-2)}{\chi^2_{p+2,\lambda}}) + 1]$$

$$= 1 + (p-2)\ e^{-\lambda/2}[\frac{\nu}{p}(\nu\phi(\frac{p}{2}-1,\frac{p}{2}+1;\ \frac{\lambda}{2})-2\phi(\frac{p}{2},\frac{p}{2}+1;\ \frac{\lambda}{2}))$$

$$+ \theta^2_i\{\frac{\nu^2(p-2)}{p(p+2)}\ \phi(\frac{p}{2},\frac{p}{2}+2;\frac{\lambda}{2}) + \frac{2\nu}{p}\ \phi(\frac{p}{2};\frac{p}{2}+1;\ \frac{\lambda}{2})$$

$$- \frac{2\nu}{p+2}\ \phi(\frac{p}{2}+1,\ \frac{p}{2}+2;\ \frac{\lambda}{2})\}].$$

An upper bound on the value of R^*_i is obtained below, from the following formulas, where prime denotes derivative with respect to y.

$$a(\phi(a+1,b;y)-\phi(a,b;y)) = y\phi'(a,b;y) = \frac{ay}{b}\ \phi(a+1,b+1;y)$$

$$(b-a) \, \Phi \, (a,b+1;y) = b\Phi(a,b;y) - b\Phi'(a,b;y).$$

Since $0 < \nu < 2$; we have

$$(2.4) \qquad \nu\Phi(\tfrac{p}{2}-1, \tfrac{p}{2}+1; \tfrac{\lambda}{2}) - 2\Phi(\tfrac{p}{2}, \tfrac{p}{2}+1; \tfrac{\lambda}{2})$$

$$< 2(\Phi(\tfrac{p}{2}-1, \tfrac{p}{2}+1; \tfrac{\lambda}{2}) - \Phi(\tfrac{p}{2}, \tfrac{p}{2}+1; \tfrac{\lambda}{2}))$$

$$= -\frac{2\lambda}{p+2} \, \Phi(\tfrac{p}{2}, \tfrac{p}{2}+2; \tfrac{\lambda}{2}).$$

The quantity inside the braces on the right side of (2.3) is by the given formulas, equal to

$$(2.5) \qquad (\frac{\nu^2(p-2)}{2p} + \frac{2\nu}{p})(\Phi(\tfrac{p}{2}, \tfrac{p}{2}+1; \tfrac{\lambda}{2}) - \Phi'(\tfrac{p}{2}, \tfrac{p}{2}+1; \tfrac{\lambda}{2}))$$

$$= \frac{\nu}{p(p+2)} \, (\nu(p-2)+4)\Phi(\tfrac{p}{2}, \tfrac{p}{2}+2; \tfrac{\lambda}{2}).$$

From (2.3), (2.4) and (2.5) we have

$$(2.6) \qquad R_i^* < 1 + \frac{\nu(p-2)}{p(p+2)} \, e^{-\lambda/2} \Phi(\tfrac{p}{2}, \tfrac{p}{2}+2; \tfrac{\lambda}{2})((\nu(p-2)+4)\theta_i^2 - 2\lambda).$$

Therefore, for $3 \le K \le p$

$$(2.7) \qquad \sum_{i=1}^{K} R_i^* < K + \frac{\nu\lambda(p-2)}{p(p+2)} \, e^{-\lambda/2}\Phi(\tfrac{p}{2}, \tfrac{p}{2}+2; \tfrac{\lambda}{2})(\nu(p-2)+4-2K)$$

$$< K = \sum_{i=1}^{K} R_i$$

for $\nu < \frac{2(K-2)}{p-2}$, where R_i denotes the contribution to the risk of X from the ith component.

Using in (2.3) the asymptotic property of the confluent hypergeometric function, given by

$$\Phi(a,b;y) = \frac{\Gamma(b)}{\Gamma(a)} e^y y^{a-b}(1+0(y^{-1})), \quad y > 0$$

we have that for large values of λ

$$(2.8) \qquad R_i^* = 1 + \frac{\nu(p-2)}{\lambda}((\nu(p-2) + 4) \frac{\theta_i^2}{\lambda} - 2) + 0(\lambda^{-2}).$$

From (2.8) it follows that $\nu < 2(K-2)/(p-2)$ is a necessary condition for the inequality (1.4) to hold uniformly in θ. Moreover, it is known that X is admissible for $p \leq 2$. Therefore, the inequality (2.7) cannot be true for $K < 3$. Hence, we have shown the following result.

<u>Theorem 2.1</u>. The inequality $\sum_{i=1}^{K} R_i^* < \sum_{i=1}^{K} R_i$ holds for all θ if and only if $3 \leq K \leq p$. and $0 < \nu < 2(K-2)/(p-2)$.

Now we consider the case when σ^2 is unknown but there is available an estimate T, say, such that $T \overset{d}{\sim} \sigma^2 \chi_m^2/m$, independent of X. In this case, substituting T for σ^2 in (1.1), Stein's estimator is given by

$$(2.9) \qquad \delta^{**}(X) = (1 - \frac{\nu(p-2)T}{S})X$$

$$= (1 - \frac{\hat{\nu}(p-2)}{S})X$$

where $\hat{\nu} = \nu T$. To compare δ^{**} with the maximum likelihood estimator X we let $\sigma = 1$, as before, without loss of generality.

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Substituting $\hat{\nu}$ for ν (1.2) and taking expectation with respect to the distribution of T we find that

$$R(\delta^{**},\theta) < R(X,\theta)$$

for $\nu<2m/(m+2)$. Similarly, we substitute $\hat{\nu}$ for ν in (2.3) and taking expectation with respect to the distribution of T, we obtain the risk of δ^{**} due to the ith component, given by

$$(2.10) \quad R_i^{**} = 1 + (p-2)e^{-\lambda/2}[\frac{\nu}{p}(\frac{\nu(m+2)}{m} \phi \ (\frac{p}{2}-1,\frac{p}{2}+1;\frac{\lambda}{2})$$

$$-2\phi(\frac{p}{2},\frac{p}{2}+1;\frac{\lambda}{2}))$$

$$+ \theta_i^2 \ \{\frac{\nu^2(m+2)(p-2)}{mp(p+2)} \phi \ (\frac{p}{2},\frac{p}{2}+2;\frac{\lambda}{2}) + \frac{2\nu}{p} \phi \ (\frac{p}{2},\frac{p}{2}+1;\frac{\lambda}{2})$$

$$- \frac{2\nu}{p+2} \phi \ (\frac{p}{2}+1, \ \frac{p}{2}+2;\frac{\lambda}{2})\}].$$

Let $\nu<2m/(m+2)$. Applying the method used in deriving (2.6) we get

$$(2.11) \quad R_i^{**} = < 1 + (p-2)e^{-\lambda/2}[\frac{2\nu}{p}(\phi(\frac{p}{2}-1,\frac{p}{2}+1;\frac{\lambda}{2})-\phi(\frac{p}{2},\frac{p}{2}+1;\frac{\lambda}{2})]$$

$$+ \theta_i^2 \ (\frac{\nu^2(m+2)(p-2)}{2mp} + \frac{2\nu}{p})(\phi(\frac{p}{2},\frac{p}{2}+1;\frac{\lambda}{2})-\phi'(\frac{p}{2},\frac{p}{2}+1;\frac{\lambda}{2})]$$

$$= 1 + \frac{\nu(p-2)}{p(p+2)} e^{-\lambda/2} \phi \ (\frac{p}{2},\frac{p}{2}+2;\frac{\lambda}{2})[\theta_i^2(\frac{\nu(m+2)(p-2)}{m} + 4)-2\lambda].$$

Therefore, for $3 \leq K \leq p$

$$(2.12) \quad \sum_{i=1}^{K} R_i^{**} < K + \frac{\nu\lambda(p-2)}{p(b+2)} e^{-\lambda/2} \phi\left(\frac{p}{2}, \frac{p}{2} + 2; \frac{\lambda}{2}\right)\left(\frac{\nu(m+2)(p-2)}{m} + 4 - 2K\right)$$

$$< K$$

for $\nu < \dfrac{2m(K-2)}{(m+2)(p-2)}$.

From (2.12) we obtain the following theorem extending the result of Theorem 2.1. The necessary part of the theorem is obtained by considering the asymptotic property of the confluent hypergeometric function as in the proof of Theorem 2.1.

Theorem 2.2. The inequality $\sum\limits_{i=1}^{K} R_i^{**} < \sum\limits_{i=1}^{K} R_i$ holds for all θ if and only if $3 \leq K \leq p$ and $0 < \nu < 2m(K-2)/(m+2)(p-2)$.

If the loss function is suitably modified then the results given above hold also for the case in which the components of X are independent but have unequal variances, as shown below. Let $X_i \overset{d}{\sim} N(\theta_i, \sigma_i^2)$, and let the loss function be given by

$$(2.13) \qquad \hat{L}(\delta, \theta) = \sum_{i=1}^{p} (\delta_i - \theta_i)^2/\sigma_i^2.$$

First suppose that the variances are known. Let $\hat{S} = \sum\limits_{i=1}^{p} X_i^2/\sigma_i^2$. Consider the estimator

$$(2.14) \qquad \hat{\delta} = (1 - \nu(p-2)/\hat{S})X$$

It is easy to see that the conclusion of Theorem 2.1 is valid

for $\hat{\delta}$ with respect to the loss given by (2.13). Next, suppose that the variances are known up to a constant factor, that is, let $\sigma_i^2 = \alpha V_i$ where α is unknown but the V_i's are known. If an estimate T is available for α, such that, $T \underset{\sim}{d} \alpha \chi_m^2/m$ then the conclusion of Theorem 2.2 is valid with respect to the loss (2.13) for the estimator.

(2.15)
$$\hat{\hat{\delta}} = (1-\nu(p-2)T/\hat{\hat{S}})X.$$

where
$$\hat{\hat{S}} = \Sigma_{i=1}^{p} X_i^2/V_i.$$

On the other hand, if the loss function is not modified, that is, we consider the original loss function equal to the sum of squared errors, then we can use the estimators $\hat{\eta}$ and $\hat{\hat{\eta}}$ for $\hat{\delta}$ and $\hat{\hat{\delta}}$, respectively, in the two cases considered in the preceding paragraph, where

(2.16)
$$\hat{\eta}_i = (1-\nu(p-2)/\sigma_i^2 S^*)X_i$$

$$\hat{\hat{\eta}}_i = (1-\nu(p-2)T/\sigma_i^2 S^*)X_i$$

where $S^* = \overset{p}{\underset{i=1}{\Sigma}} X_i^2/\sigma_i^4$. It follows from Theorem 1 of Berger (1976) that $\hat{\eta}$ dominates the maximum likelihood estimator X. By direct computation we have for the contribution to the risk of $\hat{\eta}$ from the ith component

(2.18)
$$\hat{R}_i = E(\hat{\eta}_i - \theta_i)^2$$

$$= \sigma_i^2 + 2\nu(p-2)E(\frac{2Y_i^2}{\sigma_i^2 V^2} - \frac{1}{V}) + \nu^2(p-2)^2 EY_i^2/\sigma_i^2 V^2$$

where $Y_i = \dfrac{X_i}{\sigma_i} \overset{d}{\sim} N(\dfrac{\theta_i}{\sigma_i}, 1)$ and $V = \overset{p}{\underset{i=1}{\Sigma}} Y_i^2/\sigma_i^2$. From (2.18) we have for $3 \leq K \leq p$

$$(2.19) \quad \overset{K}{\underset{i=1}{\Sigma}} \hat{R}_i \leq \overset{K}{\underset{i=1}{\Sigma}} \sigma_i^2 + 2\nu(p-2)E(\dfrac{2}{V} - \dfrac{K}{V}) + \nu^2(p-2)^2 E\dfrac{1}{V}$$

$$< \overset{K}{\underset{i=1}{\Sigma}} \sigma_i^2 = \overset{K}{\underset{i=1}{\Sigma}} R_i$$

for $0 < \nu < \dfrac{2(K-2)}{p-2}$. Similarly, for the risk of $\hat{\hat{\eta}}$ we have

$$(2.20) \quad \overset{K}{\underset{i=1}{\Sigma}} \hat{\hat{R}}_i < \overset{K}{\underset{i=1}{\Sigma}} R_i .$$

for $0 < \nu < \dfrac{2m(K-2)}{(m+2)(p-2)}$. It is easily shown that the sufficient conditions given above for the inequalities (2.19) and (2.20) are also necessary. Therefore, the conclusions of Theorem 2.1 and Theorem 2.2 hold for the estimators $\hat{\eta}$ and $\hat{\hat{\eta}}$, respectively.

Remark: A wide class of estimators for the mean of a multivariate normal distribution is known (see e.g. Bock (1975)) to dominate the maximum likelihood estimator. It would be interesting to investigate whether some of these estimators have the component-wide minimax property of Stein's estimator, shown above. It is also interesting to investigate similar property for the estimator δ^+ , obtained by substituting for the quantity inside the paranthesis on the right side of (1.1) by its positive part, the estimator δ^+ is known to dominate δ^*.

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For estimating the mean of a p-variate normal distribution it is known that Stein's rule is minimax. It is shown for a subfamily of Stein's estimators that the vector consisting of any $K \geq 3$ components of the estimator is also minimax for estimating corresponding components of the mean vector.

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